Module 2: Trigonometry

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Module 2, Section 1

Trigonometric Functions

Introduction

You have studied trigonometric ratios since Grade 9 Mathematics. In this module you will study those ratios as *functions*. You will look at tables of values of these functions, generate their graphs, and list their properties. It is important for you to do all the questions in the assignments because some new concepts are developed within the assignments. Using techniques that you developed in Module 1, you will study transformations of these circular functions. In order to find the zeros of the functions, you will develop skills in circular function equation solving and in relationships amongst these functions which are called *identities*.

Section 1 — Outline

Lesson 1 The Trigonometric Values for θ, $0° \leq \theta \leq 360°$

Lesson 2 Solving Trigonometric Equations for $0° \leq \theta \leq 360°$

Lesson 3 Special Trigonometric Values

Lesson 4 The Unit Circle

Lesson 5 The Circular Functions

Lesson 6 Radian Measures

Lesson 7 Sine and Cosine Graphs

Review

Notes

Lesson 1

The Trigonometric Values for θ, $0° \le \theta \le 360°$

To this point, you have determined the values of the trigonometric ratios for angles between 0° and 180°. In this short introductory lesson, we will explore the trigonometric ratios for any angle θ between 0° and 360°. First we will re-define the primary trig ratios sine, cosine, and tangent.

Suppose θ is any angle in standard position (its initial arm lies on the positive x-axis). Further, let P(x,y) be any point on θ's terminal arm and r be the distance from P to the origin (0,0). Since r is a distance, **it will always be considered positive**.

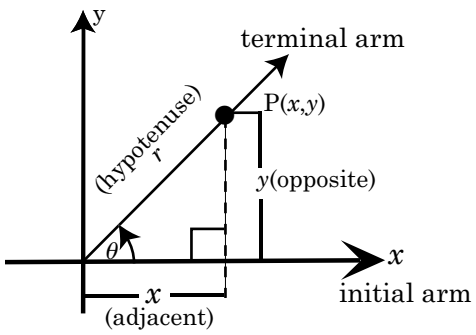
Now, using Pythagorean Theorem, we can determine the value of r:

$r = \sqrt{x^2 + y^2}$ and we define:

$$\sin\theta = \frac{y}{r} = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{x}{r} = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{y}{x} = \frac{opposite}{adjacent}$$



These definitions can be applied to determine the trigonometric ratios for any angle, θ, in standard position if the coordinates of a point on its terminal arm are known.

Example 1

Determine the three primary trig ratios for the angle θ in standard position, with the point P(–4,5) on its terminal arm.

Solution

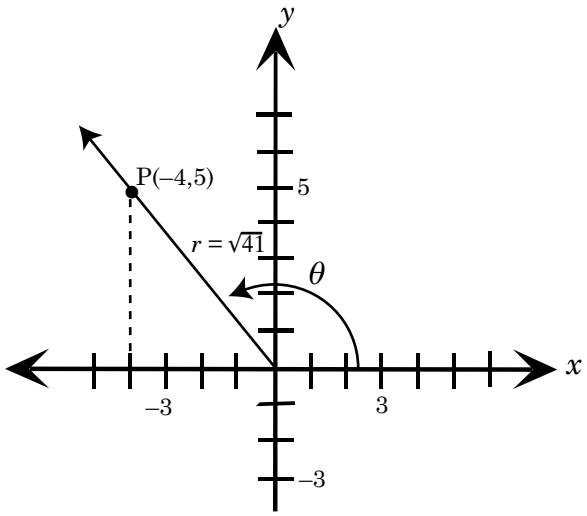
θ terminates in Quadrant II and we will be using our new definitions to solve this problem, but first we need to determine r.

$$r = \sqrt{(-4)^2 + (5)^2} = \sqrt{41}$$

$$\sin\theta = \frac{y}{r} = \frac{5}{\sqrt{41}} \approx 0.781$$

$$\cos\theta = \frac{x}{r} = \frac{-4}{\sqrt{41}} \approx -0.625$$

$$\tan\theta = \frac{y}{x} = \frac{5}{-4} = -1.250$$



As you can see, for any angle θ terminating in **Quadrant II**, as in Example 1, the only one of the three primary trig ratios which is positive is ***sin*** θ. We will come back to this result shortly.

Example 2

An angle θ in standard position has the point Q(–12,–5) on its terminal arm. Determine its primary trigonometric ratios.

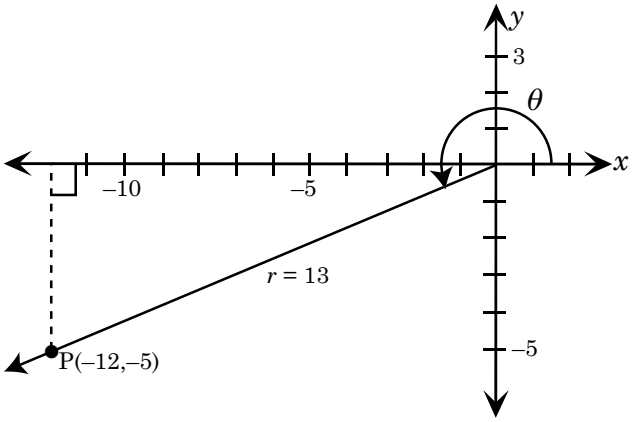
For θ in Quadrant III:

$$r = \sqrt{(-12)^2 + (-5)^2} = 13$$

$$\sin\theta = \frac{y}{r} = \frac{-5}{13} \approx -0.385$$

$$\cos\theta = \frac{x}{r} = \frac{-12}{13} \approx -0.923$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-12} = +0.417$$



Note that the only positive one of the primary trigonometric ratios for an angle θ terminating in **Quadrant III** is ***tan*** θ.

If we examine the signs of our x and y-coordinates in the other two quadrants, we see the following.

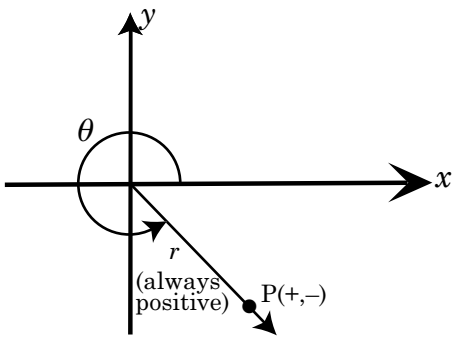
Quadrant IV

In Quadrant IV, x is positive and y is negative.

$$\text{So } \sin\theta = \frac{y}{r} = \frac{(-)}{(+)} = (-)$$

$$\cos\theta = \frac{x}{r} = \frac{(+)}{(+)} = (+)$$

$$\tan\theta = \frac{y}{x} = \frac{(-)}{(+)} = (-)$$



So for θ terminating in **Quadrant IV** only ***cos*** θ is positive.

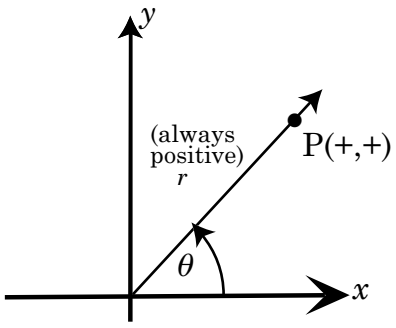
Quadrant I

In Quadrant I, x is positive and y is positive.

$$\text{So } \sin\theta = \frac{y}{r} = \frac{(+)}{(+)} = (+)$$

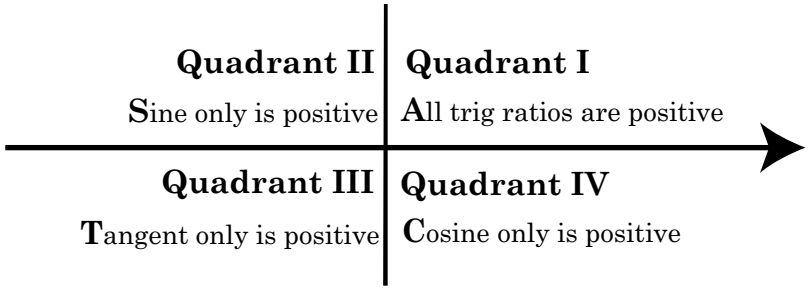
$$\cos\theta = \frac{x}{r} = \frac{(+)}{(+)} = (+)$$

$$\tan\theta = \frac{y}{x} = \frac{(+)}{(+)} = (+)$$



So for θ terminating in **Quadrant I**, **all** three of the primary trigonometric ratios are positive.

Now we are able to summarize what we know about the trig ratios of any angle in standard position.



From the first letter of each sentence, this is called the "CAST" diagram and it will prove very helpful to you in the future!

Now for a little practice.

Guided Practice

Using the new definitions for sin θ, cos θ, and tan θ, determine the exact values for these trigonometric ratios. Please leave your answers in fractional and/or radical form if they are not integer values.

1. P(1,11) is on the terminal arm.
2. Q(–6,8) is on the terminal arm.
3. R(5,–12) is on the terminal arm.
4. S(–6,–3) is on the terminal arm.
5. M(5,0) is on the terminal arm.
6. N(0,4) is on the terminal arm.
7. L(–3,0) is on the terminal arm.
8. K(0,–2) is on the terminal arm.

Check your answers in the Module 2 Answer Key.

Lesson 2

Solving Trigonometric Equations, $0° \le \theta \le 360°$

Outcomes

Upon completing this lesson, you will be able to:

- determine the three primary trigonometric ratios for any angle θ, in standard position $0° \le \theta \le 360°$
- solve trigonometric equations for $0° \le \theta < 360°$

Overview

In Lesson 1 we learned new definitions for $\sin\theta$, $\cos\theta$, and $\tan\theta$. Using these new definitions, we were able to determine the trigonometric ratios for angles terminating in any one of the four quadrants and make observations about their signs (the "CAST" diagram below).

S	A
T	C

In this lesson we will be using the CAST diagram along with a given angle's *reference angle* (defined below) to solve simple trigonometric equations.

Definition:

The ***reference angle*** for any angle θ in standard position is the **acute** angle between its terminal arm and the *nearest* part of the ***x*-axis**.

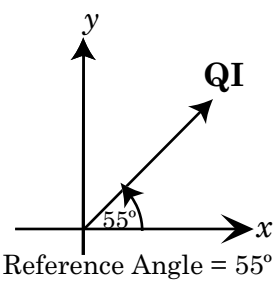
Example 1

Determine the quadrant where each of the following terminate and the reference angle.

a) 55° b) 169° c) 228° d) 310°

Solution

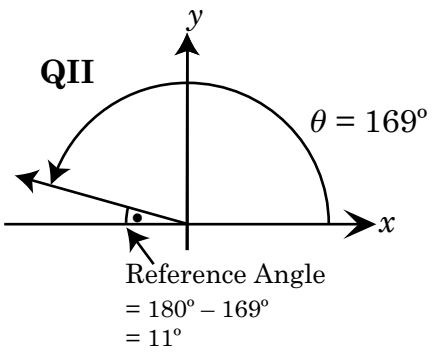
a)



Reference Angle = 55°

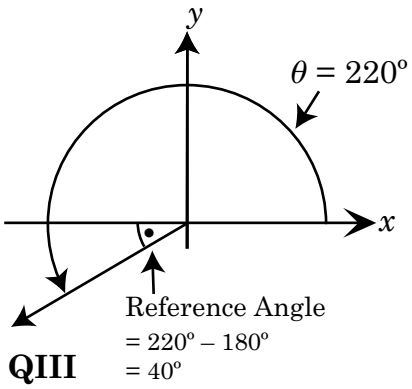
Angles which terminate in QI (Quadrant 1) are identical in measure to their reference angle.

b)



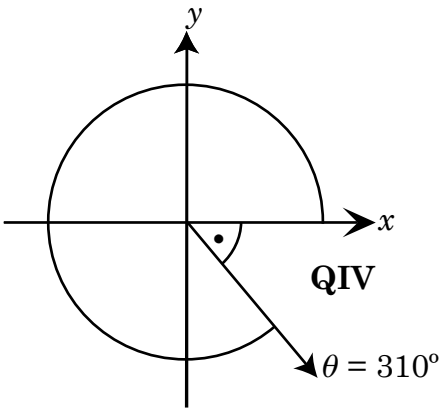
The reference angle for θ terminating in QII is given by: $180° - \theta$. This is because the nearest part of the x-axis to the terminal arm of θ is the negative x-axis, where the angle is 180° from the initial arm.

c)



The reference angle for θ terminating in QIII is given by: $\theta - 180°$. θ comes before 180 to make the reference angle positive.

d)



- $\theta_R = 360° - 310° = 50°$

  $\sin 310° = -0.766$

  $\cos 310° = 0.643$

  $\tan 310° = -1.192$

The reference angle for θ terminating in QIV is given by: $360° - \theta$.

Note: our abbreviation for "reference angle" is θ_R.

Angles having the same reference angles have very similar trigonometric ratios. The differences are in the signs (positive or negative) of each.

Definition: **Coterminal angles** are angles with the same terminal arm. Coterminal angles have the same reference angle

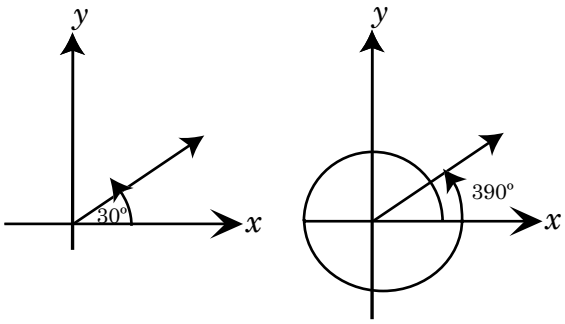
Example 2

Find an angle coterminal with:

a) 30° b) 220°

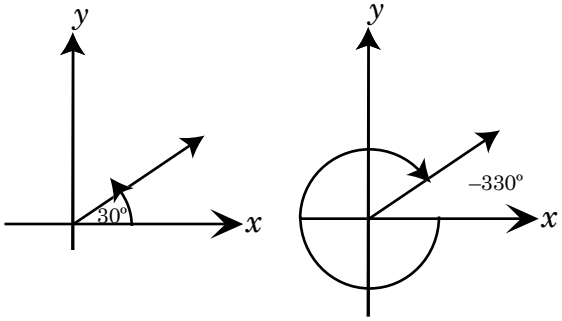
Solution

a)



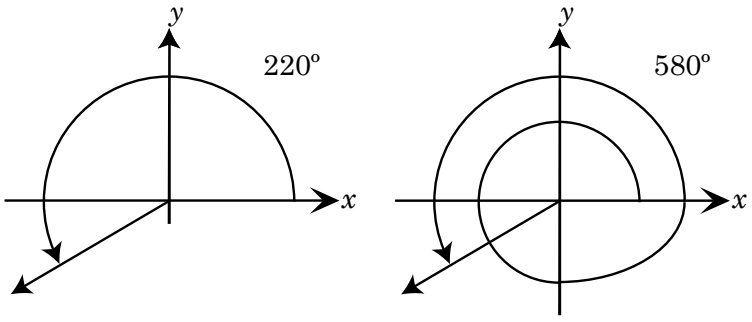
30° and 390° are coterminal because they both end up in the same position

or



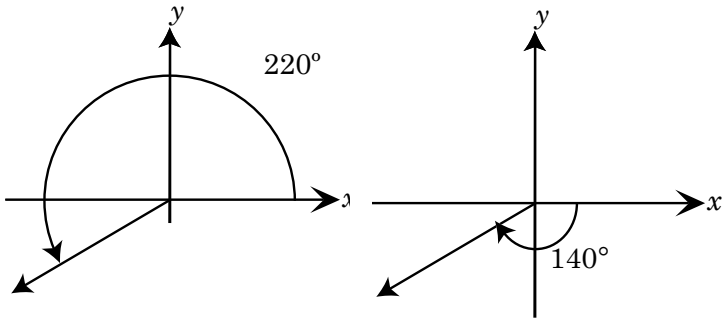
30° and –330° are coterminal because they both end up in the same position

b)



220° and 580° are coterminal because they both end up in the same position

or



220° and –140° are coterminal because they both end up in the same position

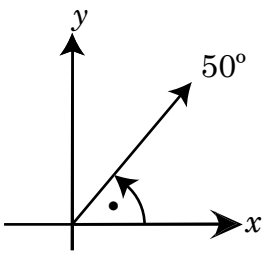
Example 3

Use your calculator to determine the three primary trigonometric ratios for each of the following:

a) 50° b) 130° c) 230° d) 310°

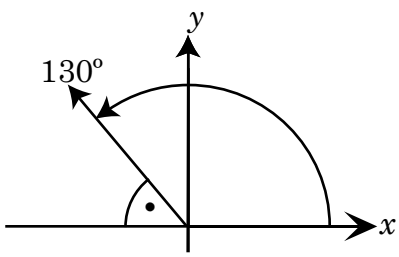
Solution

a)



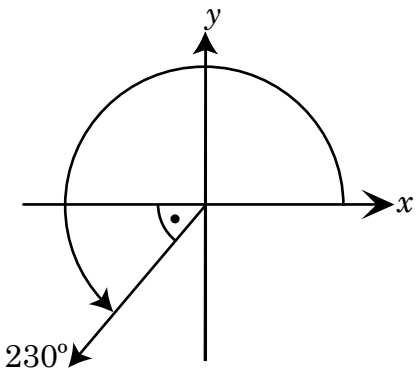
- $\theta_R = 50°$
  sin 50° = 0.766
  cos 50° = 0.643
  tan 50° = +1.192

b)



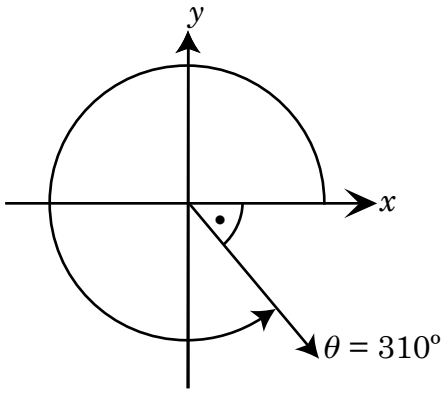
- $\theta_R = 180° - 130° = 50°$
  sin 130° = 0.766
  cos 130° = –0.643
  tan 130° = –1.192

c)



- $\theta_R = 230° - 180° = 50°$
  sin 230° = –0.766
  cos 230° = –0.643
  tan 230° = 1.192

d)



- $\theta_R = 360° - 310° = 50°$
  sin 310° = –0.766
  cos 310° = 0.643
  tan 310° = –1.192

You can check these results against the CAST diagram, too. In Quadrant I(a) all trig ratios are positive, in Quadrant II (b) only sin is positive, in Quadrant III (c) only tan is positive, and in Quadrant IV (d) only cos is positive.

Now we will turn things around a little and solve some equations for θ, $0° \leq \theta < 360°$.

Example 4

Solve for θ if $\sin\theta - 0.528 = 0$. Express θ to the nearest whole degree.

$0° \le \theta < 360°$

Solution

If $\sin\theta - 0.528 = 0$

$\sin\theta = 0.528$ (positive)

Now using the "CAST" diagram below . . .

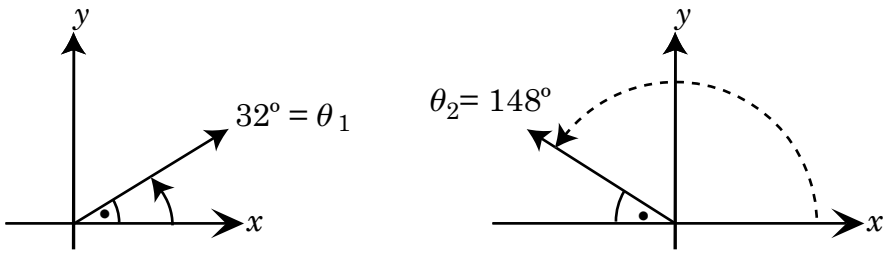
S	A
T	C

we see that sine is positive in both Quadrants I and II, so there are *two* solutions; θ, in Quadrant I and θ_2 in Quadrant II. Both θ_1 and θ_2 will have the same reference angle which we will determine first.

$\theta_R = \sin^{-1}(0.528)$ [Calculator]

$\doteq 32°$. This is the reference angle.

$\therefore \theta_1 = 32°$ and $\theta_2 = 180° - 32° = 148°$



Example 5

Solve for θ if $12\cos\theta + 5 = 2\cos\theta$, $0° \le \theta < 360°$

Solution

Rearrange the equation first.

$10\cos = -5$

$\cos\theta = \dfrac{-5}{10} = -0.500$ (negative)

S	A
T	C

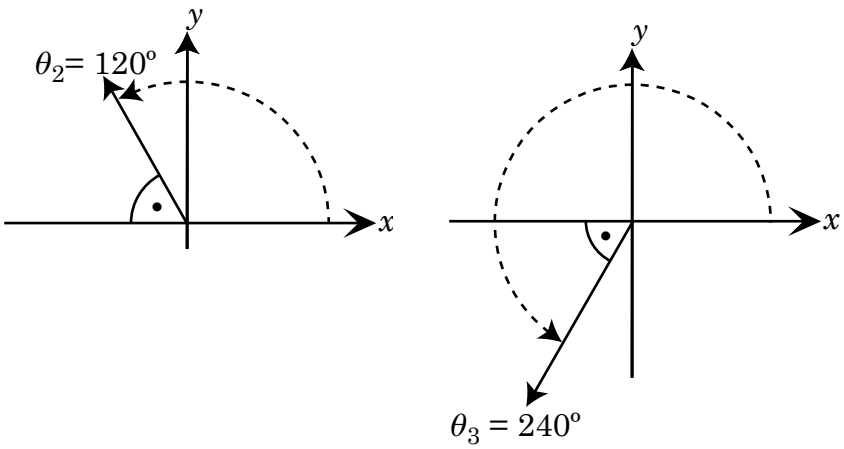
If cosine is negative, then θ must terminate in Quadrant II or Quadrant III. (Using CAST we see that cosine is positive in the other two Quadrants.)

Now $\theta_R = \cos^{-1}(0.500)$

- $\theta_R = 60°$

$\therefore \theta_2 = 180° - 60° = 120°$ $\theta_3 = 180° + 60° = 240°$

[We ignore signs when determining the reference angle because reference angles always lie between 0° and 90°.]



Example 6

Solve for θ: $\frac{3\tan\theta - 4}{2} = 1$ $0° \le \theta < 360°$. Answer to nearest degree.

Solution

Rearrange the equation first.

$$\frac{3\tan\theta - 4}{2} = 1$$
$$3\tan\theta - 4 = 2$$
$$3\tan\theta = 6$$
$$\tan\theta = 2.000 \text{ (positive)}$$

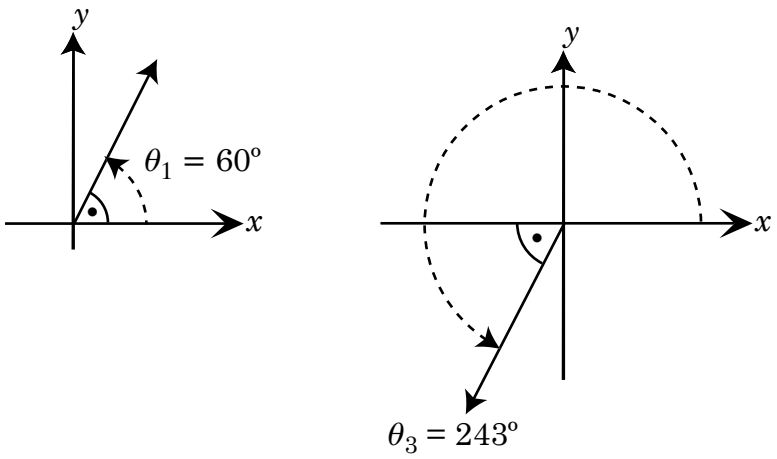
S	A
T	C

tan θ positive means that θ terminates in Quadrant I or Quadrant III. (Using CAST.)

- $\theta_R = \tan^{-1}(2.000)$

$\doteq 63°$

$\theta_1 = 63°$ and $\theta_3 = 180° + 63° = 243°$



Example 7

Solve for θ if $2\cos^2\theta + 3\cos\theta - 2 = 0$
$0^\circ \le \theta < 360^\circ$

Solution

If $2\cos^2\theta + 3\cos\theta - 2 = 0$

$(2\cos\theta - 1)(\cos\theta + 2) = 0$ [Factoring the Trinomial]

$\therefore \cos\theta = \frac{1}{2}$ or $\cos\theta = -2$

$\cos\theta = 0.500$ or $\cos\theta = -2$

Case 1:

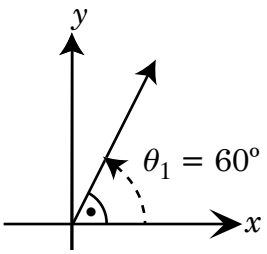
If $\cos\theta = 0.500$ (positive)

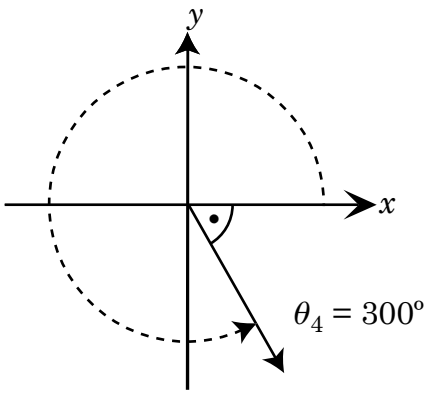
S	A
T	C

then θ terminates in Quadrant I or Quadrant IV.

- $\theta_R = \cos^{-1}(0.500) = 60°$

$\therefore \theta_1 = 60°$ or $\theta_4 = 360° - 60° = 300°$





Case 2:

$\cos\theta = -2$

- $\theta_R = \cos^{-1}(-2)$ No solution

This is because $\sin\theta$ and $\cos\theta$ range between -1 and 1.

So the solution set in Example 5 is {60°, 300°}.

Now it's your turn to try a few.

Guided Practice

Solve for θ, $0° \le \theta < 360°$, in each of the following. Answer to the nearest degree, or state an exact solution where possible.

1. a) Find the reference angles for 116°, 265°, 289°, 323°.

   b) Evaluate the three trigonometric ratios for the angles above.

   c) Find a coterminal angle with the angles above.

2. $3 \sin \theta = -2$

3. $5 \tan \theta + 4 = -2 \tan \theta$

4. $3 \sin \theta + 1 = 0$

5. $3 \cos^2\theta + 14 \cos \theta = 5$

6. $7 \cos \theta - 8 = 4$

7. $2 \sin^2 \theta = \sqrt{3} \sin \theta$

8. $4 \tan^2 x - 13 \tan x = 12$

Check your answers in the Module 2 Answer Key.

Lesson 3

Special Trigonometric Values

Outcomes

Upon completing this lesson, you will be able to:

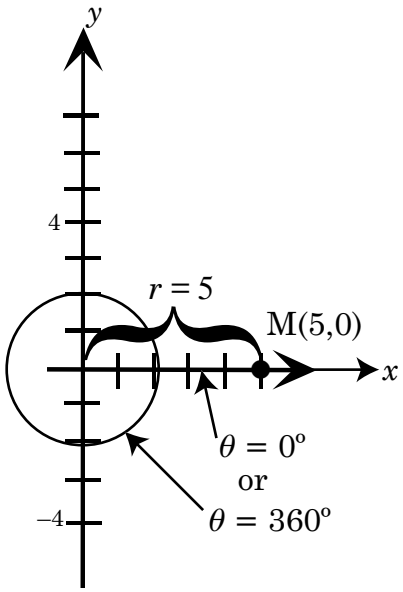
- state the trigonometric ratios for any "Quadrantal" angle θ, $0° \leq \theta \leq 360°$
- state the trigonometric ratios of any multiple of 30°, 45°, or 60° without the use of a calculator or trigonometric table
- solve linear and quadratic trigonometric equations with solutions which are multiples of 30°, 45°, or 60° (these equations will be solved without the use of any aids)

Overview

In Lesson 2 you determined the trigonometric ratios for any θ, where $0° \leq \theta \leq 360°$. You also solved various trigonometric equations for θ, $0° \leq \theta < 360°$, using a calculator and the CAST diagram. In this lesson you will be solving the same type of problems except *without* the use of a calculator. This will require some recall of Lesson 1 and the establishment of some geometric facts.

In the Lesson 1, Guided Practice 5–8, you determined the three primary trigonometric ratios for four very special angles which we call **Quadrantal**. They are angles which terminate on the positive or negative x or y-axis, thereby dividing the coordinate plane into the four quadrants. Let's look at them once more so that we can summarize the results.

0°, 360°



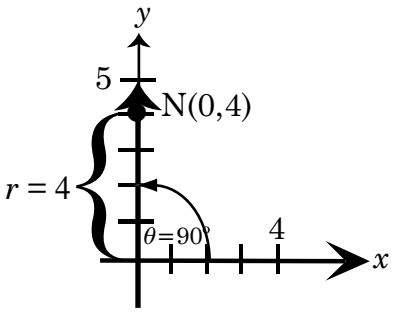
The point (5,0) is on the terminal arm of the angle θ, which represents a rotation of either 0° or 360° from its initial arm. Therefore:

$$\sin 0° = \sin 360° \qquad \frac{y}{r} = \frac{0}{5} = 0$$

$$\cos 0° = \cos 360° \qquad \frac{x}{r} = \frac{5}{5} = 1$$

$$\tan 0° = \tan 360° \qquad \frac{y}{x} = \frac{0}{5} = 0$$

90°



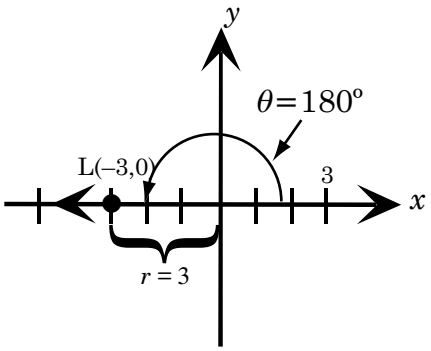
The point (0,4) is on the terminal arm of the angle θ, which represents a rotation of 90° from its initial arm. Therefore:

$$\sin 90° = \frac{y}{r} = \frac{4}{4} = 1$$

$$\cos 90° = \frac{x}{r} = \frac{0}{4} = 0$$

$$\tan 90° = \frac{y}{x} = \frac{4}{0} \text{ undefined}$$

180°



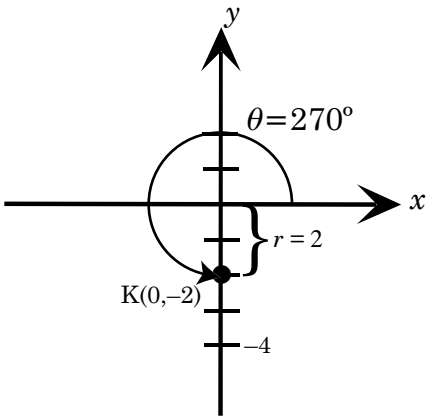
The point (–3,0) is on the terminal arm of the angle θ, which represents a rotation of 180° from its initial arm. Therefore:

$$\sin 180° = \frac{y}{r} = \frac{0}{3} = 0$$

$$\cos 180° = \frac{x}{r} = \frac{-3}{3} = -1$$

$$\tan 180° = \frac{y}{x} = \frac{0}{-3} = 0$$

270°



The point (0,–2) is on the terminal arm of the angle θ, which represents a rotation of 270° from its initial arm. Therefore:

$$\sin 270° = \frac{y}{r} = \frac{-2}{2} = -1$$

$$\cos 270° = \frac{x}{r} = \frac{0}{2} = 1$$

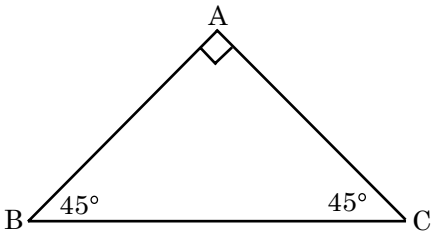
$$\tan 270° = \frac{y}{x} = \frac{-2}{0} \text{ undefined}$$

The following table summarizes the trigonometric values for the Quadrantal angles.

θ	0°/360°	90°	180°	270°
$\sin\theta$	0	1	0	–1
$\cos\theta$	1	0	–1	0
$\tan\theta$	0	∞	0	∞

Now for some geometric facts.

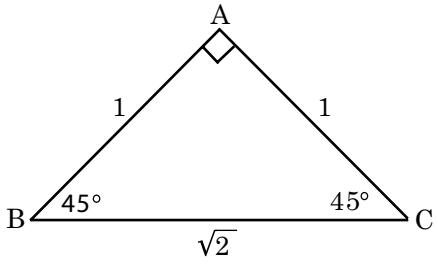
Fact 1: 45°- 45°- 90° Triangle



Since Δ ABC is isosceles it follows that AB = AC. Let these two equal lengths be x and use the Pythagorean Theorem to find the length of side BC.

$$(\text{BC})^2 = (\text{AB})^2 + (\text{AC})^2$$
$$(\text{BC})^2 = (x)^2 + (x)^2$$
$$(\text{BC})^2 = 2(x)^2$$
$$\text{BC} = x\sqrt{2}$$

Therefore, the ratios of a 45°- 45°- 90° triangle are always $x : x : x\sqrt{2}$, or $1 : 1 : \sqrt{2}$, respectively. This fact enables you to find the trigonometric ratios of a 45° angle.



Therefore,

$$\sin 45° = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{1}{1} = 1$$

You should memorize these ratios. When you are answering a question based on the 45°-45°-90° triangle, you may find that sketching the above reference triangle, with its ratios, is useful.

Example 1

Without the use of a calculator:

a) simplify $\sin 45°(\cos 45°) - \tan 45°$

b) solve for θ, if $0° \le \theta \le 360°$: $\sin\theta = \sqrt{2}\sin^2\theta$

Solution

a) $\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$

b) $\sin\theta - \sqrt{2}\sin^2\theta = 0$

$\sin\theta\left(1 - \sqrt{2}\sin\theta\right) = 0$ (factor)

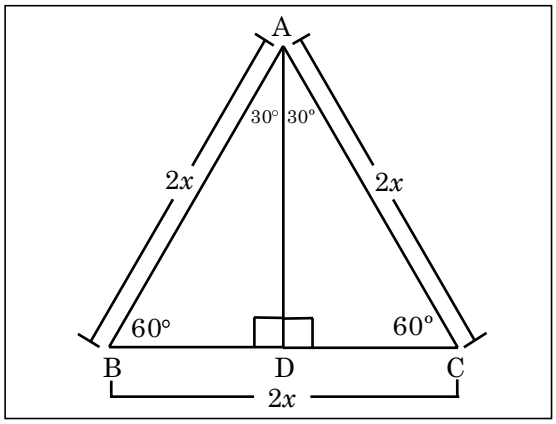
$\therefore$ Either $\sin\theta = 0$ or $\sin\theta = \frac{1}{\sqrt{2}}$

$\sin\theta$ is positive in both cases, so θ lies in Quadrants 1 and 2 only.

$\theta = 0°,\ 180°,\ 45°,\ 135°$

Fact 2: 30°-60°-90° Triangle

Beginning with an equilateral Δ ABC each side of length $2x$, you form two 30°-60°-90° triangles by dropping a perpendicular, AD, from vertex A.



Triangles ABD and ACD are congruent by ASA. Hence,

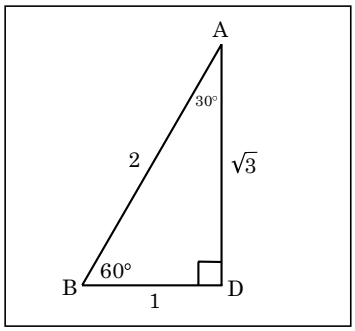
$\text{BD} = \frac{1}{2}\text{BC} = x.$

Note: (ASA: Angle-Side-Angle: the two triangles have equal 30° angles, equal 60° angles, and sides of equal length 2x which lie between the two angle(s).

You use the Pythagorean Theorem to find the third side AD.

$$(\text{AD})^2 = (\text{AB})^2 - (\text{BD})^2$$
$$(\text{AD})^2 = (2x)^2 - (x)^2$$
$$(\text{AD})^2 = 3x^2$$
$$\text{AD} = x\sqrt{3}$$

Therefore, the ratios of a 30°-60°-90° triangle are always $x : x\sqrt{3} : 2x$, or $1 : \sqrt{3} : 2$, respectively. This fact enables you to find the trigonometric ratios of a 30° or a 60° angle.



$$\sin 30° = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{1}{2} \qquad \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2}$$

$$\tan 30° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{1}{\sqrt{3}} \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

You should memorize these values! When you are answering a question based on a 30° - 60° - 90° triangle, you may find that sketching the above reference triangle, with its ratios, is useful. Observe the length of the longest side—it's 2, not √3. Many students overlook that and put √3 on the wrong side of the triangle.

Example 2

Without the use of a calculator:

a) find the value of $2\cos 30° + \sin 60° \tan 30°$

b) solve for x if $0° \le x < 360°$: $\sin x \cdot \tan x - \sqrt{3} \sin x = 0$

Solution

a) $2\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{3}}\right)=\sqrt{3}+\frac{1}{2}=\frac{2\sqrt{3}+1}{2}$

b) $\sin x\left(\tan x-\sqrt{3}\right)=0$

$\sin x=0 \text{ or } \tan x-\sqrt{3}=0$

$\sin x=0 \text{ or } \tan x=\sqrt{3}$

$x=0°, 180°, 60°, \left(180+60\right)°$

$x=0°, 180°, 60°, 240°$

The following activity should serve as a review of your previous trigonometry experiences. The only new work is that you are to try to do this work without using a calculator.

Guided Practice

1. Complete the following table with exact values. Do as much as you can without a calculator. You will be required to be able to state these values without a calculator, whenever the question asks for ***exact values***.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$									
$\cos\theta$									
$\tan\theta$									

	210°	225°	240°	270°	300°	315°	330°	360°
$\sin\theta$								
$\cos\theta$								
$\tan\theta$								

2. Determine the ***exact*** value for each expression.
   a) $\cos 45° \sin 45° - \sin 60°$
   b) $\tan 30° \tan 60° + \sin 90°$
   c) $\sin^2 120° + \cos^2 120°$ $\left(\text{Recall that } \sin^2 x = \left(\sin x\right)^2.\right)$

d) $\dfrac{\tan 60^\circ}{\sin 60^\circ} - \cos^2 135^\circ$

e) $\sin 210^\circ \cos 210^\circ \tan^2 210^\circ$

f) $\cos^2 150^\circ - \sin^2 150^\circ + \cos 300^\circ$

g) $\sin 150^\circ \cos 60^\circ + \cos 150^\circ \sin 60^\circ - \sin 210^\circ$

3. Find the value(s) of θ, where $0° < \theta < 360°$, that satisfy the following equations:

a) $\sin\theta = \dfrac{1}{\sqrt{2}}$

b) $\tan\theta = \sqrt{3}$

c) $\cos\theta = \dfrac{\sqrt{3}}{2}$

d) $\cos\theta = -\dfrac{1}{2}$ and $\tan\theta > 0$

e) $\sin\theta = \dfrac{\sqrt{3}}{2}$ and $\cos\theta < 0$

f) $2\cos\theta = 2$

g) $2\sin^2\theta + \sin\theta = 0$

h) $4\cos^2\theta = 1$

i) $2\tan\theta + 2\sqrt{3} = 0$

j) $2\sin^2\theta - \sin\theta = 1$

k) $\cos^2\theta - \cos\theta = 0$

Check your answers in the Module 2 Answer Key.

Lesson 4

The Unit Circle

Outcomes

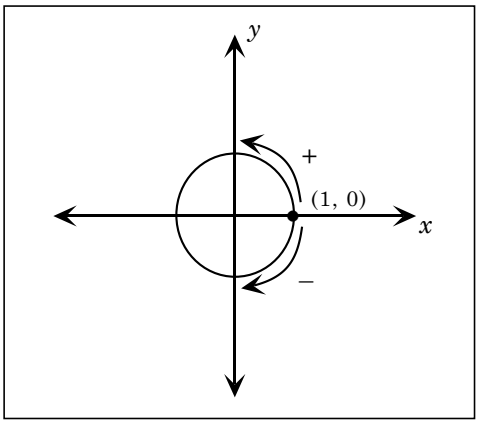
Upon completing this lesson, you will be able to:

- locate the position of the terminal point P(θ) on the unit circle for special values of the real number θ
- state a value of θ which would produce a given position of a special circular point P(θ)

Overview

This lesson is a lead-in to the study of circular functions. However, your ability to locate special points on the unit circle is extremely important for the rest of this module.

Definition: A ***unit circle*** is the circle with centre at the origin and with a one-unit radius.



The circumference of this circle is $2\pi r = 2\pi(1) = 2\pi$.

We now set up a circular function game in which you move certain distances around this circle governed by the following rules:

1. The point with coordinates (1,0) is the starting point for all of your trips around the circle. It corresponds to the *initial arm* defined in Lesson 1. The arc is then said to be in ***standard position***.
2. If the distance to be moved is positive you will move in a counterclockwise direction along the unit circle.
3. If the distance to be moved is negative you will move in a clockwise direction along the unit circle.

4. The notation P(θ) will be used to indicate where your trip has terminated after you have moved a distance of θ units. P(θ) will be called the circular, or terminal, point.

We use θ to represent a distance here rather than an angle, but do not be concerned. We will connect this distance to an angle soon!

The following diagram will be used to label some special points on the unit circle so that you can try some trips:

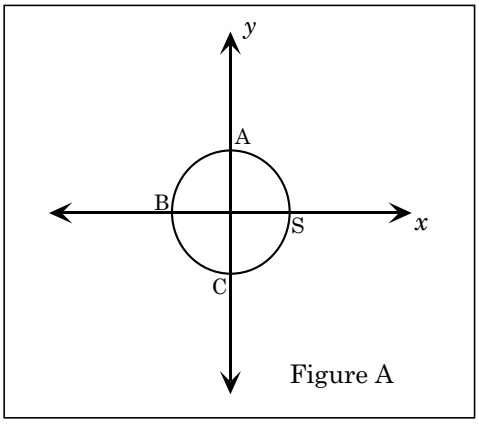


Figure A

Where is P(0)? This notation asks, "What is your terminal point if you do not move from the starting point?"

Answer

S, since you always begin your trips from the point with coordinates (1,0).

Where is P(π)? This asks for the circular point of a counterclockwise trip halfway around the circle.

Answer

B

Note: Since the circumference of the unit circle is 2π it follows that the distance:

once around the unit circle is $\boldsymbol{2\pi}$

Therefore, the distance halfway around the circle must be π.

Many students feel that going once around the circle should be π. Well it is ***not***! You may wish to remember the following fact:

halfway around the unit circle is $\boldsymbol{\pi}$

Exercise 1

Use Figure A (previous page) to fill in the table.

θ	$-\pi$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	-2π	$\frac{5\pi}{2}$	$-\frac{7\pi}{2}$	2π	101π	100π
P(θ)											

Solution

θ	$-\pi$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	-2π	$\frac{5\pi}{2}$	$-\frac{7\pi}{2}$	2π	101π	100π
P(θ)	B	A	C	C	A	S	A	A	S	B	S

To make the game more interesting, four additional points are inserted at the midpoints of each quarter arc.

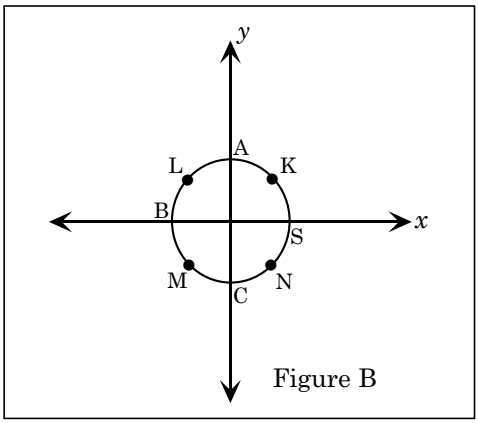
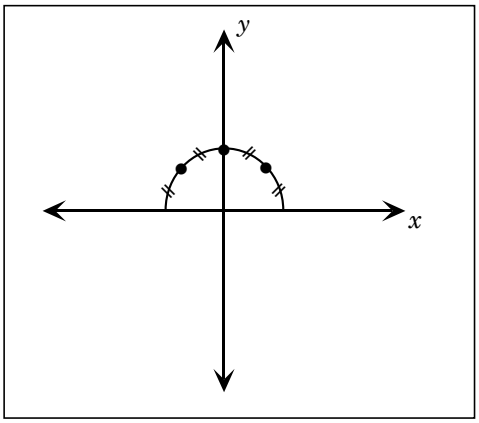


Figure B

The midpoints subdivide a semi-circle of arc length π into four equal arcs, each of length $\frac{\pi}{4}$.



Exercise 2

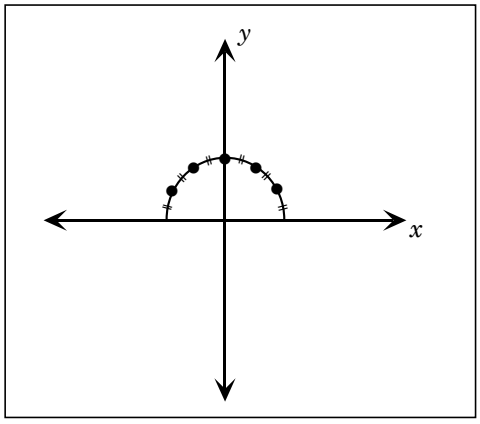
Use Figure B to fill in the table.

θ	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{5\pi}{4}$	$-\frac{7\pi}{4}$	$\frac{11\pi}{4}$	$-\frac{57\pi}{4}$	$\frac{83\pi}{4}$
P(θ)											

Solution

θ	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{5\pi}{4}$	$-\frac{7\pi}{4}$	$\frac{11\pi}{4}$	$-\frac{57\pi}{4}$	$\frac{83\pi}{4}$
P(θ)	K	L	N	M	L	N	M	K	L	N	L

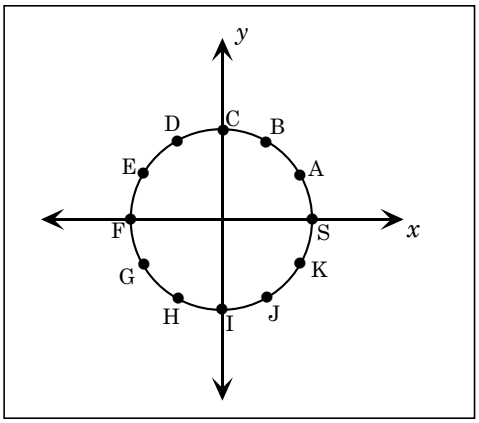
To make the game still more interesting, we subdivide the semicircle into six arcs of equal length by trisecting each of the quarter circles.



Each small arc has a length of $\frac{\pi}{6}$. Since $\frac{2\pi}{6}$ reduces to $\frac{\pi}{3}$, it follows that you will have multiples of $\frac{\pi}{6}$ and multiples of $\frac{\pi}{3}$ as you travel around the circle.

Exercise 3

Use the diagram below to fill in the table. (Assume that all the little arcs are equal.)



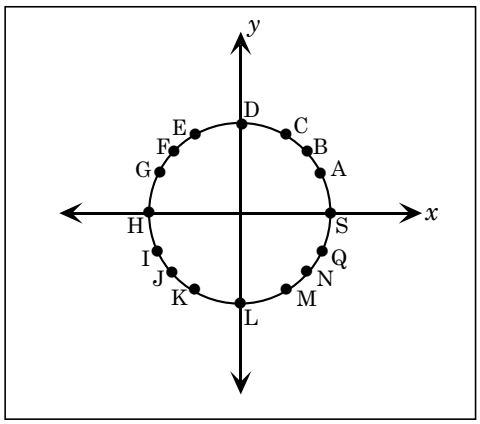
θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$-\frac{\pi}{3}$	$-\frac{2\pi}{3}$	$-\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{3}$	$-\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$	$-\frac{13\pi}{6}$
$P(\theta)$											

Solution

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$-\frac{\pi}{3}$	$-\frac{2\pi}{3}$	$-\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{3}$	$-\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$	$-\frac{13\pi}{6}$
$P(\theta)$	A	B	J	H	G	G	J	E	K	B	K

Guided Practice

If we include the midpoints and the trisection points on one unit circle, the following diagram results. Use this diagram to answer the questions in the assignment.



1. Find the letter corresponding to the following value of θ:

a) $\frac{2\pi}{3}$ b) $\frac{7\pi}{6}$

c) 27π d) $\frac{5\pi}{4}$

e) $-\frac{\pi}{6}$ f) 14π

g) $-\frac{5\pi}{2}$ h) 53π

i) $-\frac{11\pi}{6}$ j) $\frac{17\pi}{3}$

k) $\frac{\pi}{2}$ l) $-\pi$

m) $\frac{4\pi}{3}$ n) $\frac{3\pi}{2}$

o) $\frac{5\pi}{6}$ p) $\frac{15\pi}{4}$

q) $-\frac{7\pi}{3}$ r) 4π

2. Find the letter(s) corresponding to the following value of θ:
   a) $k\pi$ if k is an even integer (e.g., 2π, -2π, etc. . . .)
   b) $k\pi$ if k is an odd integer (e.g., -1π, 1π, 3π, etc. . . .)
   c) $\frac{k\pi}{2}$ if k is an integer (e.g., $\frac{-\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}$, etc....)

3. Find the letter(s) corresponding to the following value of θ:
   a) $\frac{k\pi}{4}$ if k is an odd integer (e.g., $\frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$, etc....)
   b) $\frac{k\pi}{4}$ if k is an even integer (e.g., $\frac{2\pi}{4} = \frac{\pi}{2}$, $\frac{4\pi}{4} = \pi$ etc. . . .)

4. State the smallest ***positive*** value of θ which will terminate at the point with letter:

a) B	b) H
c) M	d) L
e) A	f) C
g) S	h) F
i) I	j) D

5. State the smallest ***negative*** value of θ which will terminate at the point with letter:

a) B	b) H
c) M	d) L
e) A	f) K
g) S	h) F
i) I	j) N

Check your answers in the Module 2 Answer Key.

Lesson 5

The Circular Functions

Outcomes

Upon completing this lesson, you will be able to:

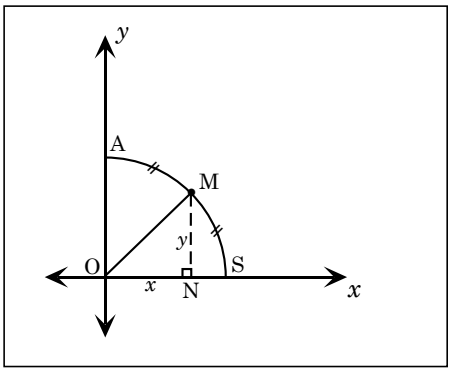
- state the coordinates of special values of the circular points P(θ)
- define the trigonometric functions as circular functions
- define the three reciprocal functions in terms of their reciprocals
- solve circular function equations, which have real number solutions

Overview

For every arc length θ on the unit circle there is only one point, P(θ), determined by this arc. Hence, you have a function which maps every arc into a unique point. Now your emphasis will switch from the point to the ***coordinates*** of the point. We redefine P(θ) as the ordered pair of numbers rather than just the location.

For example, P(0) = (1,0), our starting point; $P\left(\frac{\pi}{2}\right) = (0,1)$, the coordinates at the top of the unit circle; and so on. For each real number θ there corresponds an ordered pair of numbers (x, y), which are the coordinates of P(θ).

The above two examples were rather easy because the circular point terminated on the axis. What happens when the point is not on an axis? We will fill in the coordinates of the special terminal points in the first quadrant and assign the rest as an exercise.



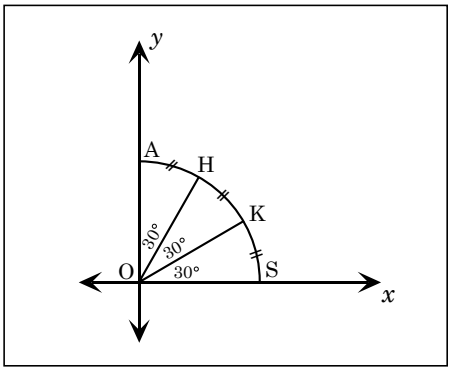
Since M is the midpoint of the arc it follows
$\angle \text{MON} = \frac{1}{2}\left(90^{\circ}\right) = 45^{\circ}.$

$\sin \angle \text{MON} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{\text{OM}} = \frac{y}{1}$ since the radius of the unit circle is one unit.

Furthermore, you know from Lesson 2 of this module that
$\sin 45^{\circ} = \frac{1}{\sqrt{2}}$. Therefore, $y = \frac{1}{\sqrt{2}}$.

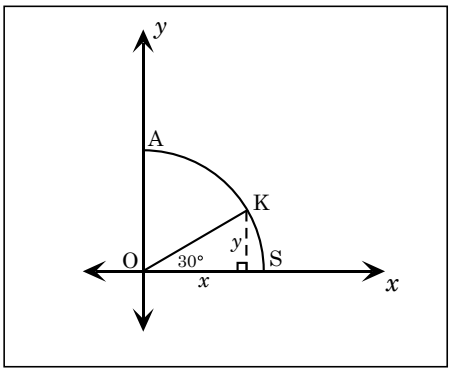
Additionally, Δ MON is isosceles, so $x = \frac{1}{\sqrt{2}}$.

Hence, $\text{P}\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ or coincidentally
$\text{P}\left(\frac{\pi}{4}\right) = \left(\cos 45^{\circ}, \sin 45^{\circ}\right)$.



If points H and K are the trisection points of arc AS, it follows that $\angle$ KOS = $\angle$ HOK = $\angle$ AOH = 30°. Therefore, $\angle$ HOS = 60°.

Let's go after the coordinates of K and H!



$\sin \angle \text{KOS} = \frac{y}{\text{OK}} = \frac{y}{1}$ since the radius of the unit circle is 1.

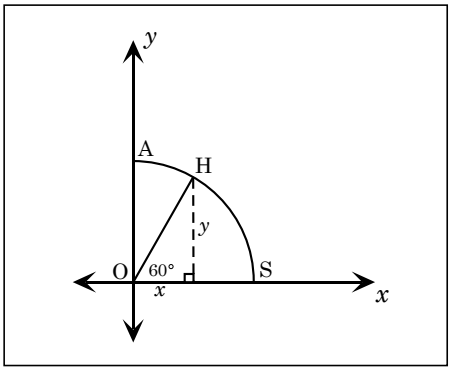
And, since $\angle \text{KOS} = 30°$ it follows $y = \sin 30° = \frac{1}{2}$ (from Lesson 2).

Similarly, $\cos 30° = \frac{x}{\text{OK}}$. Therefore, $x = \cos 30° = \frac{\sqrt{3}}{2}$.

Hence, $\text{P}\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ or $\text{P}\left(\frac{\pi}{6}\right) = (\cos 30°,\ \sin 30°)$.

You can use the figure below to conclude that:

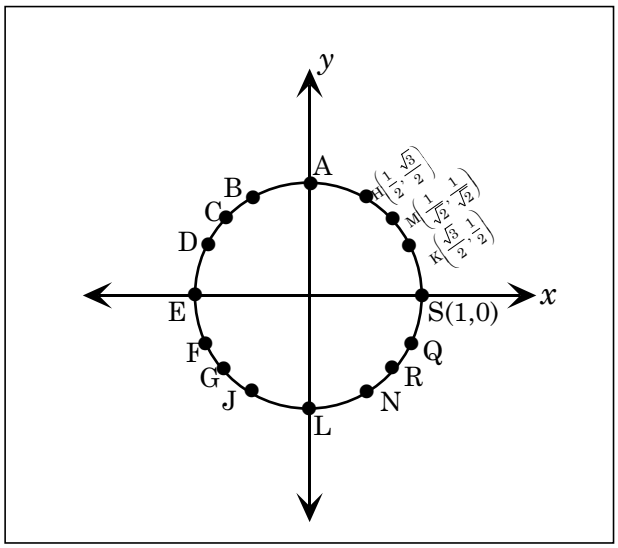
$$\text{P}\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ or } \text{P}\left(\frac{\pi}{3}\right) = (\cos 60°,\ \sin 60°)$$



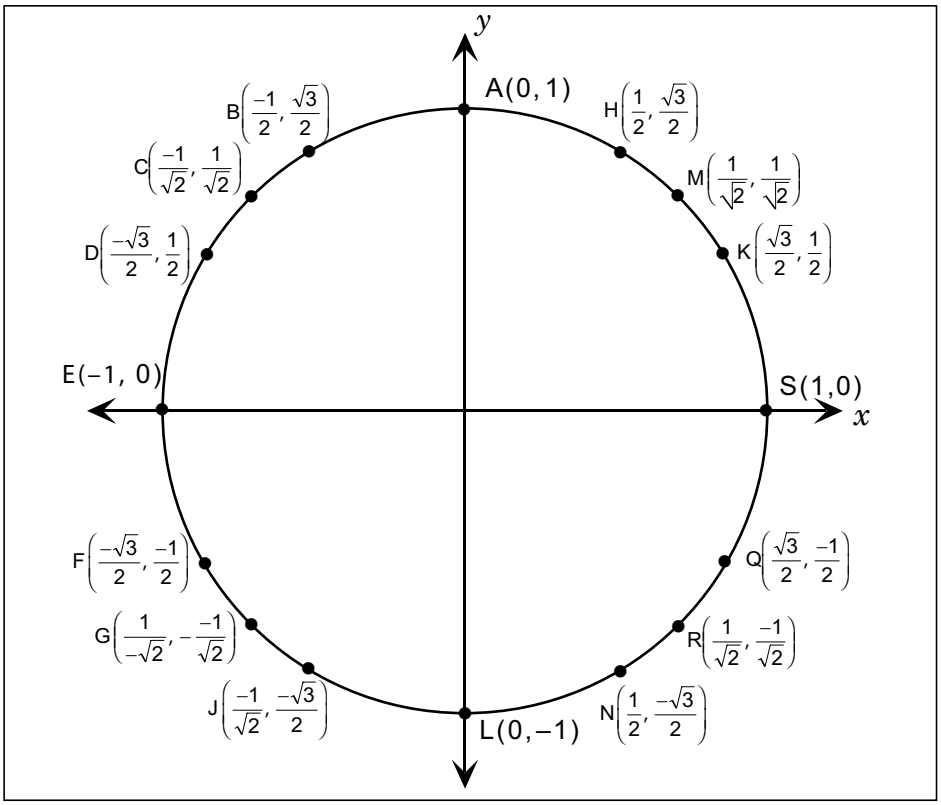
At this point, by using the symmetry of a circle, you are capable of completing the coordinates of every special point on the unit circle.

Exercise 1

Use the stated coordinates to complete the coordinates of every other labeled special circular point.



Solution



Note: You should know the coordinates of these points by heart, or at least those in Quadrant I—then you can use that information along with the CAST diagram to quickly determine the other points.

As mentioned previously, we have a function which matches a real number θ with an ordered ***pair*** of numbers (x, y). Mathe-maticians prefer to match each real number with only *one* real number. This is accomplished by taking a different approach to the trigonometric ratios, an approach which was suggested by realizing that the coordinates of the points on the unit circle are generated by trigonometric function values.

Alternate definitions of trigonometric ratios (These are also called circular functions.)

If θ is an arc in standard position (this means on a circle of radius 1) and $P(\theta) = (x, y)$ is the terminal point for the real number θ, then:

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \cos\theta = \frac{x}{r} = \frac{x}{1} = x \quad \text{and}$$

$$\tan\theta = \frac{y}{x}, \text{ where } x \neq 0 \quad \left[\text{Since } \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

Example 1

Use the coordinates on the unit circle and the definitions of the circular functions to find:

a) $\sin\frac{\pi}{6}$ b) $\cos\pi$ c) $\tan\left(\frac{-\pi}{3}\right)$

d) $\sin\frac{\pi}{4}$ e) $\cos\frac{\pi}{6}$ f) $\sin^2\frac{\pi}{4} + 2\cos\frac{\pi}{6}$

Solution

a) $\frac{1}{2}$

This is a good sequence of steps to reason through:

1. Locate where the point terminates on the unit circle. (K in this case)
2. Think of the coordinates of this point. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
3. Since $\sin\theta = y$, select the second coordinate as your answer: $\frac{1}{2}$

b) -1

This is a good sequence of steps to reason through:

1. Locate where the point terminates on the unit circle: E
2. Think of the coordinates of this point: (–1,0)
3. Since $\cos\theta = x$, select the first coordinate as your answer.

c) $-\sqrt{3}$

This is a good sequence of steps to reason through:

1. Locate where the point terminates on the unit circle.
2. Think of the coordinates of this point.
3. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x} = \dfrac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

d) $\sin\dfrac{\pi}{4}$

1. arc terminates at M
2. coordinates of M $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$
3. Since $\sin\theta = y = \dfrac{1}{\sqrt{2}}$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

e) $\cos\dfrac{\pi}{6}$:

1. arc terminates at K
2. coordinates of K $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$
3. Since $\cos\theta = x = \dfrac{\sqrt{3}}{2}$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

f) $\sin^2\dfrac{\pi}{4} + 2\cos\dfrac{\pi}{6} = \left(\dfrac{1}{\sqrt{2}}\right)^2 + 2\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{1}{2} + \dfrac{2\sqrt{3}}{2} = \dfrac{2\sqrt{3}+1}{2}$

Now we'll go backwards.

Example 2

Solve the following over the interval $0 \le \theta < 2\pi$. Give solutions as exact values.

a) $\cos\theta = \dfrac{-\sqrt{3}}{2}$

b) $2\sin^2\theta - \sin\theta = 1$

Solution

a) $\cos\theta = \dfrac{-\sqrt{3}}{2}$

$\cos\theta = x$. The x-coordinate is equal to $\dfrac{-\sqrt{3}}{2}$

in two locations: $\mathrm{D}\left(\dfrac{-\sqrt{3}}{2}, \dfrac{1}{2}\right)$ and $\mathrm{F}\left(\dfrac{-\sqrt{3}}{2}, \dfrac{-1}{2}\right)$

$\mathrm{D}\left(\dfrac{-\sqrt{3}}{2}, \dfrac{1}{2}\right) = \mathrm{P}\left(\dfrac{5\pi}{6}\right)$ and $\mathrm{F}\left(\dfrac{-\sqrt{3}}{2}, \dfrac{-1}{2}\right) = \mathrm{P}\left(\dfrac{7\pi}{6}\right)$

$\therefore$ *Solutions*: $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

b) If $2\sin^2\theta - \sin\theta = 1$

then $2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta + 1)(\sin - 1) = 0$

$2\sin\theta + 1 = 0 \quad \text{or} \quad \sin\theta - 1 = 0$

$\sin\theta = \dfrac{-1}{2} \quad \text{or} \quad \sin\theta = 1$

$\sin\theta = y$. The y-coordinate is equal to $\dfrac{-1}{2}$ or 1

$\mathrm{P}\left(\dfrac{7\pi}{6}\right) = \mathrm{F}\left(\dfrac{-\sqrt{3}}{2}, \dfrac{-1}{2}\right)$ and $\mathrm{P}\left(\dfrac{11\pi}{6}\right) = \mathrm{Q}\left(\dfrac{\sqrt{3}}{2}, \dfrac{-1}{2}\right)$

$\mathrm{P}\left(\dfrac{\pi}{2}\right) = \mathrm{A}(0,1)$

$\therefore$ *Solutions*: $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{\pi}{2}$

There are three more circular functions defined as reciprocals of the basic three. They are called the secant (written as sec), cosecant (written as csc), and cotangent (written as cot) functions.

Definition:

If θ is an arc length in standard position, with terminal circular point P(θ), then:

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \text{ or } \frac{1}{\tan\theta}$$

provided the denominators are not equal to zero.

Example 3

Determine the exact value of

$$\sec^2\frac{\pi}{6} - 3\cot\frac{\pi}{4}$$

Solution

$$\sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \rightarrow \sec^2\frac{\pi}{6} = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

$$\cot\frac{\pi}{4} = \frac{1}{\tan\frac{\pi}{4}} = \frac{1}{1} = 1 \rightarrow 3\cot\frac{\pi}{4} = 3(1) = 3$$

$$\therefore \sec^2\frac{\pi}{6} - 3\cot\frac{\pi}{4} = \frac{4}{3} - 3 = \frac{4}{3} - \frac{9}{3} = \frac{-5}{3}$$

Example 4

Solve for θ (exactly) for $0 \le \theta < 2\pi$: $\csc^2\theta + \csc\theta = 0$

Solution

If $\csc^2\theta + \csc\theta = 0$

$\csc\theta(\csc\theta + 1) = 0$

$\therefore \quad \csc\theta = 0 \qquad$ or $\csc\theta + 1 = 0$

$\sin\theta = \frac{1}{0} \qquad \csc\theta = -1$

(no solution) $\qquad \sin\theta = \frac{1}{-1} = -1 = y$

$P\left(\frac{3\pi}{2}\right) = L(0,-1)$

$\therefore \theta = \frac{3\pi}{2}$

The Link Between Angles in S.P. and Coordinates on Unit Circle

The following diagram shows the link between the angle in standard position (or the length) in the unit circle and the coordinates of the corresponding point on the unit circle.

Given θ in standard position, $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$

and because $r = 1$ in the unit circle, it is simple to use this diagram to find exact trig. function values of any angle which terminates on any of the points shown.

Examples

$\sin\frac{3\pi}{4} = \frac{y}{r} = \frac{1}{\sqrt{2}}$

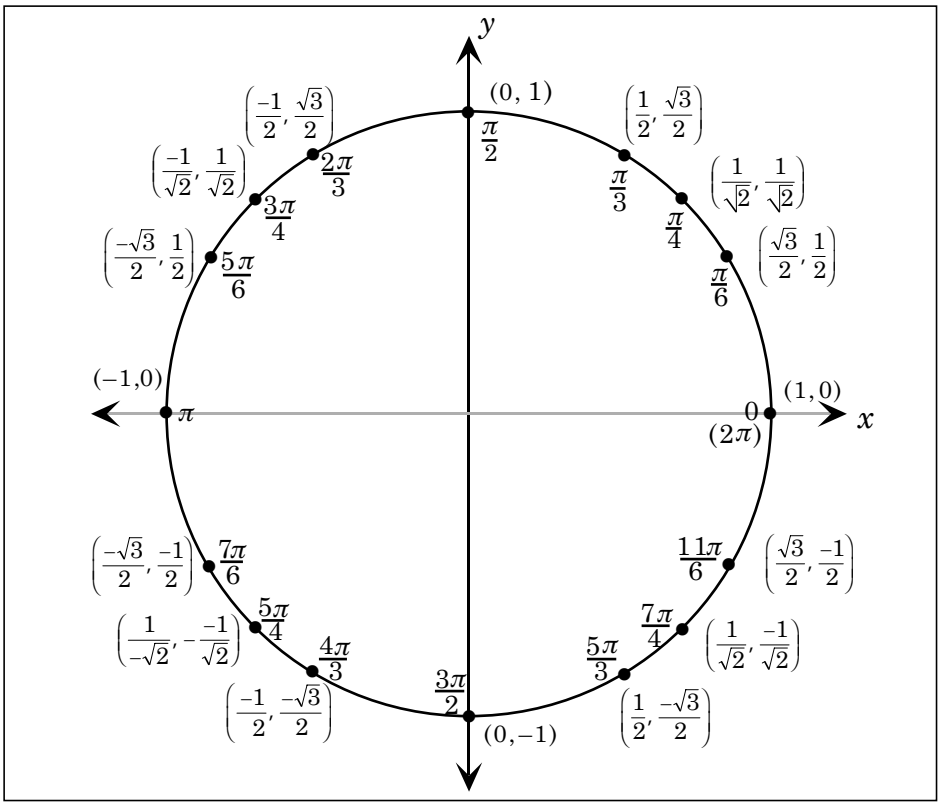
$\cos\frac{11\pi}{6} = \frac{x}{r} = \frac{\sqrt{3}}{2}$

$\tan\frac{4\pi}{3} = \frac{y}{x} = \frac{-\sqrt{3}}{2} \Big/ \frac{-1}{2}$

$= \sqrt{3}$

Also $\sec\frac{2\pi}{3} = \frac{1}{\cos\frac{2\pi}{3}} = \frac{r}{x} = -2$

$\cot\frac{3\pi}{2} = \frac{1}{\tan\frac{3\pi}{2}} = \frac{x}{y} = \frac{0}{-1} = 0$



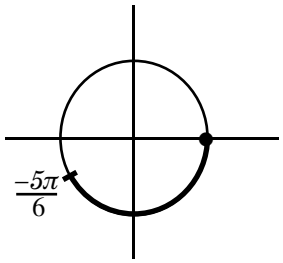
For angles outside the interval $(0, 2\pi)$, we can still locate the corresponding point on the unit circle.

Example:

Find the exact value of $\csc\left(\frac{-5\pi}{6}\right)$.

Solution

$\frac{-5\pi}{6}$ is the arc of the unit circle obtained by starting at the positive x-axis and travelling CLOCKWISE round the unit circle for a distance of $\frac{-5\pi}{6}$.



This is the same point on the unit circle we would have obtained by travelling COUNTER CLOCKWISE from the positve x-axis for the rest of the circumference of the circle, i.e.,

$$2\pi - \frac{5\pi}{6}$$
$$= \frac{12\pi}{6} - \frac{5\pi}{6}$$
$$= \frac{7\pi}{6}$$

Thus
$$\csc\left(\frac{-5\pi}{6}\right)$$
$$= \csc\left(2\pi - \frac{5\pi}{6}\right)$$
$$= \csc\frac{7\pi}{6}$$

Guided Practice

This activity is almost identical to the assignment at the end of the previous two lessons except you will be using arc lengths instead of angle measures.

1. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for each of the following values of θ. In each question, take the extra step to write the coordinates of P(θ).

a) $\frac{2\pi}{3}$	b) $\frac{7\pi}{6}$	c) 27π
d) $\frac{5\pi}{4}$	e) $-\frac{\pi}{6}$	f) 14π
g) $-\frac{5\pi}{2}$	h) 53π	i) $-\frac{11\pi}{6}$
j) $\frac{17\pi}{3}$	k) $\frac{\pi}{2}$	l) $-\pi$
m) $\frac{4\pi}{3}$	n) $\frac{3\pi}{2}$	o) $\frac{5\pi}{6}$
p) $\frac{15\pi}{4}$	q) $-\frac{7\pi}{3}$	r) 4π

2. Determine the ***exact*** value of each expression.

Note: Although each expression involves functions and should contain notation such as sin (π), mathematicians always like shortcuts and as long as there is no ambiguity, they write these circular function statements as sin π, etc. This convention is used throughout this course.

a) $2\cos\frac{\pi}{3}+\tan\frac{\pi}{4}$

b) $\sin\left(-\frac{15\pi}{2}\right)\cos 20\pi\tan\frac{13\pi}{6}$

c) $\tan^2\frac{\pi}{6}$

d) $\cos(\sin\pi)$

e) $\tan\frac{2\pi}{3}\cos\left(\frac{-5\pi}{4}\right)+\sin\frac{3\pi}{2}\tan\frac{5\pi}{6}$

f) $\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$

g) $\sin \frac{2\pi}{3} \cos \frac{7\pi}{6} \tan \left(\frac{-3\pi}{4}\right)$

h) $\sin \left(\frac{-47\pi}{2}\right) \cos \left(-47\pi\right)$

3. Solve the following equations over the interval $0 \le \theta < 2\pi$. Do these questions without the use of a calculator. All answers must be ***exact*** special values.

a) $\sin \theta = \frac{1}{\sqrt{2}}$

b) $\tan \theta = \sqrt{3}$

c) $\cos \theta = \frac{\sqrt{3}}{2}$

d) $\cos \theta = -\frac{1}{2}$ and $\tan \theta > 0$

e) $\sin \theta = \frac{\sqrt{3}}{2}$ and $\cos \theta < 0$

f) $2 \cos \theta = 2$

g) $2 \sin^2 \theta + \sin \theta = 0$

h) $4 \cos^2 \theta = 1$

i) $2 \tan \theta + 2\sqrt{3} = 0$

j) $2 \sin^2 \theta - \sin \theta = 1$

k) $\cos^2 \theta - \cos \theta = 0$

4. Determine the ***exact*** value of each expression.

a) $\sec \frac{\pi}{3} \sin \frac{\pi}{6} - \cot \frac{\pi}{4}$

b) $\sec^2 \frac{\pi}{6} + \csc^2 \frac{\pi}{6}$

c) $\cot^2 \frac{5\pi}{6}$

d) $\tan \frac{2\pi}{3} \cot \frac{2\pi}{3}$

e) $\cos^2 \frac{\pi}{4} \sec^2 \frac{\pi}{4}$

f) $4 \csc \frac{-7\pi}{6}$

5. Find the ***exact*** solution of the following equations over the interval $0 \le \theta < 2\pi$.

a) $\sec \theta = 2$

b) $\csc \theta = \sqrt{2}$

c) $\cot \theta = -1$, and $\sin \theta < 0$

d) $\cot \theta - \cot \theta \sin \theta = 0$

e) $\tan^2 \theta - 1 = 0$

f) $\csc \theta - \sin \theta = 0$.

Check your answers in the Module 2 Answer Key.

Lesson 6

Radian Measures

Outcomes

Upon completing this lesson, you will be able to:

- convert an angle from degrees to radians and vice versa
- convert an angle from degrees to revolutions and vice versa
- convert an angle from revolutions to radians and vice versa
- solve angle measure problems, using any of the three units of measure
- relate arc length and central angles using the formula $s = \theta r$

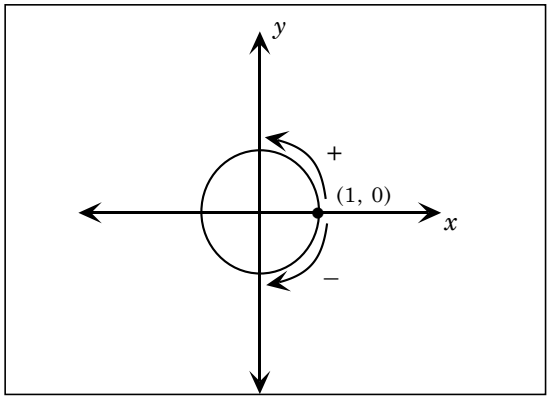
Overview

You should have noticed in the last lesson that a relationship seems to exist between the arcs on the unit circle and the trigonometric values of the central angles (angles with vertices at the centre). You will develop this relationship in this lesson.

You are familiar with ***degree*** measure where:

one revolution = 360° (once around the circle)

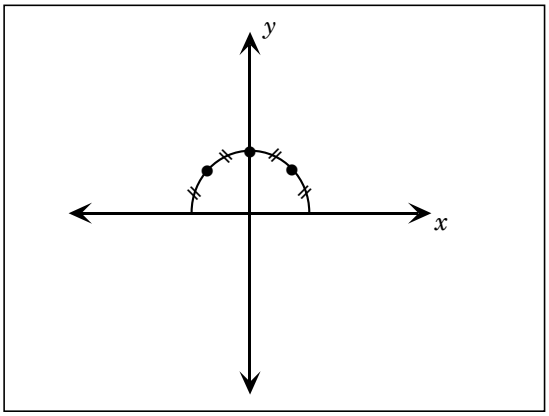
A second method of measuring angles is in terms of ***radians***.
Definition: When the arc of a circle has the same length as the radius of the circle, then the measure of the angle, with vertex at the centre, subtended by this arc is ***1 radian***.



Circle centre O

$\angle$ AOB = 1 radian

Thus, for an arc of length s units, the radian measure of the corresponding central angle is given by the number of radii needed to make up the length of the arc s.



$$O = \frac{s}{r} \text{ radians}$$

For example, an arc of length 10 cm in a circle of radius 4 cm has a central angle whose measure is given by

$$O = \frac{s}{r}$$
$$= \frac{10}{4}$$
$$= 2.5 \text{ radians}$$

Exercise 1

Use the definition of radian measure to fill in the following table.

Mathematical agreement: If no symbol appears after an angle measure, the measure of the angle is in radians. To remind people that the measure is in radians you can also use a superscript r to denote radian measure. Hence, the following are all equivalent:

$$x = x \text{ radians} = x^r$$

Arc Length	Radius	Central Angle (Radians)
2	2	
6	3	
20	5	
2	4	
16	4	

Solution

Arc Length	Radius	Central Angle (Radians)
2	2	1
6	3	2
20	5	4
2	4	0.5
16	4	4

The relationship among these three quantities is apparent. The central angle is equal to the ratio of the arc length to the radius. Stated in a product form:

arc length = central angle in radian measure × radius

Many physics books use the symbol s for distance and θ for a angle, so the following formula for this relationship has endured:

$s = \theta r$, where s is the arc length

θ is the measure of the central angle in radians

r is radius

Example 1

a) Find the length of the arc subtending by a central angle of 3 if the radius of the circle is 20 cm.

b) Find the length of the radius of a circle if a two-radian central angle is subtended by a 15 m arc.

Solution

a) Since $s = \theta r$, it follows that $s = 3(20) = 60$ cm.

b) Since $s = \theta r$, it follows that $15 = 2(r)$ so that $r = 7.5$ m.

Furthermore, since the circumference of a circle is $2\pi r$, it follows that:

Using the formula

$$s = \theta r$$

$$2\pi r = \theta r$$

$$2\pi = \theta$$

So one revolution is 2π radians.

One revolution = $2\pi r$ and since

one revolution = 360° it follows that

$360° = 2\pi$ radians.

Exercise 2

a) Convert 1° into radians.

b) Convert 1 radian into degrees.

This exercise provides you with a method to easily convert degrees to radians and radians to degrees.

Method

Since $360° = 2\pi$ radians, by dividing by 360,

$$\frac{360°}{360} = \frac{2\pi}{360} = \frac{\pi}{180}$$

$1^{\circ} = \frac{\pi}{180}$ radians (about 0.0175 radians if you do the division)

Since $1^{\circ} = \frac{\pi}{180}$ radians, to convert x degrees to radians

multiply x by $\frac{\pi}{180}$.

Since $360^{\circ} = 2\pi$ radians, by dividing by 2π,

$\frac{360°}{2\pi} = 1$ radian

$\left(\frac{180°}{\pi}\right) = 1$ radian, so 1 radian is about 57.3 degrees.

Since $1 = \left(\frac{180°}{\pi}\right)$, to convert x radians to degrees

multiply x by $\frac{180}{\pi}$.

Example 2

a) Convert 45° into radians.

b) Convert π radians into degrees.

Solution

a) $45^\circ = 45\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$ radians

b) $\pi = \pi\left(\frac{180}{\pi}\right) = 180^\circ$

You are now in a position to link trigonometric ratios and circular function values.

Let's look at the circular function values for central angles given in radians.

Example 3

Determine the value (exact if possible, otherwise to three decimal places) of:

a) $\sin\left(\frac{\pi}{4}\right)^r$ (Small superscript "r" means "radians.")

b) $\cos 3^r$

Solution

a) $\sin\left(\frac{\pi}{4}\right)^r = \sin\frac{\pi^r}{4}\left(\frac{180}{\pi}\right)^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}$ (45° is one of our "special" angles)

b) $\cos 3^r = \cos 3\left(\frac{180}{\pi}\right)^\circ = \cos\left(\frac{540}{\pi}\right)^\circ \approx -0.990$ (calculator)

Example 4

Find the exact value, if possible; otherwise use a calculator to find the approximate value.

a) $\cos \pi$ b) $\sin 20^\circ$

c) $\tan \frac{3\pi}{4}$ d) $\tan 1$

Solution

a) π is a special value so the memorized answer is $\cos \pi = -1$.

b) Since 20° is not a special value, use a calculator, in degree mode, to find sin 20° = 0.34202.

c) Special value, hence $\tan \frac{3\pi}{4} = -1$.

d) Not a special value. Use the calculator, in radian mode, to find tan 1 = 1.55741.

Note: it was not 1° so it must mean 1 radian.

Unless stated otherwise, you should always attempt to find the trigonometric, or circular function, value without a calculator. If the measure is not a special value, or special value multiple, use a calculator—use radian mode for arcs or angles in radian measure, and degree mode for angles stated in degree measure. To check if your graphing calculator is in radian mode or in degree mode, press MODE. To change the mode, move the cursor to the preferred mode (Radian or Degree) and press ENTER.

Example 5

Convert each angle measure into revolutions.

a) $\frac{5\pi^r}{6}$ b) 210°

Solution

a) If 1 revolution = 2πr

and x revolutions $= \frac{5\pi^r}{6}$

then $\frac{1}{x} = \frac{2\pi}{\frac{5\pi^r}{6}} \rightarrow 2\pi x = \frac{5\pi}{6} \rightarrow x = \frac{5\pi}{6} \times \frac{1}{2\pi} = \frac{5}{12}$ revolutions

b) If 1 revolution = 360°

and x revolutions $= 210^o$

then $\frac{1}{x} = \frac{360}{210} = \frac{12}{7} \rightarrow 12x = 7 \rightarrow x = \frac{7}{12}$ revolutions

When we solve an equation such as $\cos 2\theta = \frac{\sqrt{3}}{2}$ for $0 \le \theta < 2\pi$,

we get not two but four solutions. Here's why.

Example 6

Solve $\cos 2\theta = \frac{\pi}{6}$ for $0 \le \theta < 2\pi$

Using the CAST diagram, 2θ is positive in Quadrants I and IV.

$2\theta = \frac{\pi}{6}$ or $2\theta = 2\pi - \frac{\pi}{6}$

$2\theta = \frac{\pi}{6}$ or $2\theta = \frac{11\pi}{6}$

$\theta = \frac{\pi}{12}$ or $\theta = \frac{11\pi}{12}$

When we divide by 2, we reduce the domain of the solutions from [0,2p) to [0,p). In order to obtain all solutions for 0 £ q < 2p, we need to travel around the unit circle twice.

Instead of $2\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$

we need $2\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$ or $2\pi + \frac{\pi}{6}$ or $2\pi + \frac{11\pi}{6}$

$= \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$, or $\frac{23\pi}{6}$

Dividing by 2, $\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}$ or $\frac{23\pi}{12}$

We now have four solutions in the domain $[0,2\pi)$.

In summary:

1. The equation $\sin 2\theta = \frac{\sqrt{3}}{2}$ would also have four solutions over the domain $[0,2\pi)$.
2. The equation $\sin 3\theta = \frac{\sqrt{3}}{2}$ would have six solutions over the domain $[0,2\pi)$.
3. If we solve trig equations in degrees instead of radians, the equation $\sin 2\theta = \frac{\sqrt{3}}{2}$ would have four solutions over the domain $[0,360°)$.

Guided Practice

1. Convert each degree measure into radian measure.

   a) 25° b) –125°

   c) 460° d) 330°

2. Convert each radian measure into degree measure. Round off to one decimal place where necessary.

   a) $\frac{-7\pi}{6}$ b) $\frac{11\pi}{12}$

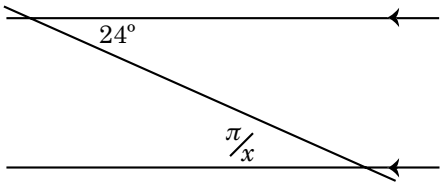
   c) 2.634 d) –0.9825

3. Determine the complement of $\frac{5\pi}{12}$. What is the supplement of this angle? (***Recall:*** Two angles are complementary if the sum of their measures is 90°. Two angles are supplementary if the sum of their measures is 180°.)

4. Express the supplement of 130° in radian measure, exactly.

5. An isosceles triangle has a base angle with measure $\frac{2\pi}{7}$. What are the measures of the other two angles? Express your answer in radians.

6. Two parallel lines have interior alternate angles with measures 24° and $\frac{\pi}{x}$. Find the value of x.



7. In what quadrant(s) are the following conditions satisfied?

   a) $\sin\theta < 0$ and $\cos\theta > 0$

   b) $\tan\theta > 0$

   c) $\sin\theta > 0$ and $\cos\theta < 0$

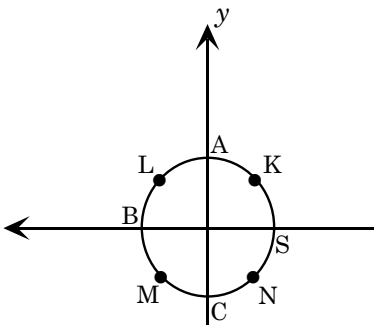
   d) $\cos\theta < 0$

Remember the CAST diagram.

S	A
T	C

8. An alien being, walking along the equator of her planet, travelled a distance of 12.3 clicks. If the diameter of this planet is 16.4 clicks, through what central angle did this alien travel? Express your answer to the nearest tenth of a degree.

9. In the diagram AC = 6 cm, BC = 8, and AB is a diameter. Find the circumference of this circle.



10. If a wheel having a circumference of 30 cm rolls 5 cm, how many radians has it turned? How many degrees has it turned?

11. Convert each measure into revolutions:
    a) 3 radians
    b) 75°

12. Use a calculator, using five digits after the decimal, in order to find each value:

a) $\tan 20$	b) $\cos 20°$
c) $\sin 2.4$	d) $\sec 1.5$
e) $\cot 3$	f) $\sin 25°$
g) $\cos\frac{7}{\pi}$	h) $\tan \pi°$
i) $\csc 5$	j) $\sin\frac{2}{\pi}$
k) $\cot 0.5$	l) $\sin(-3)$
m) $\cos(-1)$	n) $\csc 4°$
o) $\tan(-2)$	

13. Use a calculator to solve the following simple equations for x, $0 \le x < 2\pi$. Round your answer to two digits after the decimal. (***Hint:*** Use $\sin^{-1}$ function on your calculator. Since x is in radians, have your calculator set to radian mode. Be sure to find all angles, not just those in Quadrant I.)

a) $\sin x = 0.123$ b) $\cos x = -0.123$

c) $\tan x = 2$ d) $\sin x = -0.25$

e) $\cos x = 0.675$ f) $\sec x = 2.5$

g) $\cot x = -4$ h) $\csc x = 6$

The following questions are very important. Attempt to do as much as you can without using a calculator. If you must use a calculator, be sure your setting is to the correct mode.

14. Solve each equation for x, where $0° \le x < 360°$.

a) $2 \sin^2 x = \sin x$ b) $\sin x = \cos x$

c) $\sin 2x = 1$ d) $2 \cos 2x = 1$

e) $2 \csc x = 4$ f) $2 \tan^2 x - 1 = \tan x$

15. Solve each equation for x, where $0 \le x < 2\pi$.

a) $\cos^2 x - 2 \cos x = 0$ b) $\tan 2x = 1$

c) $\csc x = \sec x$ d) $2 \sin(x - \pi) - 1 = 0$

e) $\cos\left(x - \frac{\pi}{2}\right) = 0$ f) $\sin x \tan x + \sin x = 0$

Check your answers in the Module 2 Answer Key.

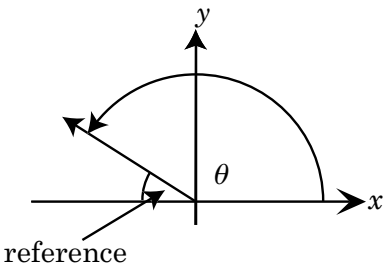
Review Section 1 before attempting the **review** questions following the summary on the next page. These questions should help you consolidate your knowledge as you prepare Section Assignment 2.1.

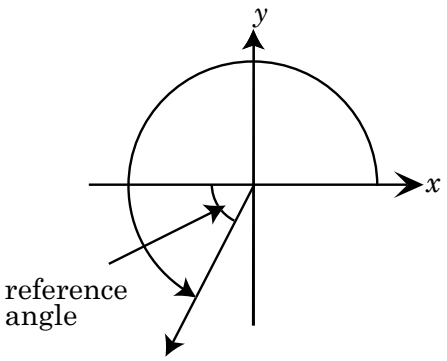
Summary of Section 1

The Basics

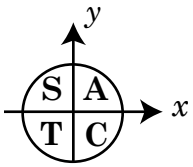
The initial arm is the positive direction of the x-axis. A positive angle is made by moving in a counter-clockwise direction from the initial arm.

A reference angle is always made with the x-axis.





CAST Diagram



This diagram indicates the quadrants where the basic trigonometric ratios are positive.

Exact Values

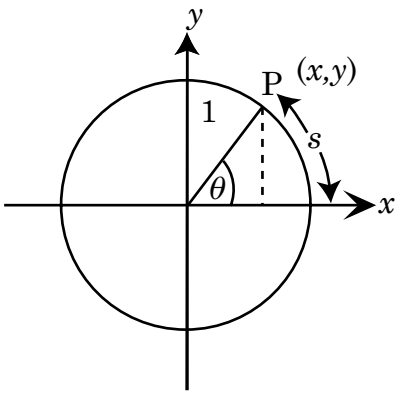
Special trigonometric values can be calculated for Quadrant I and from there the values can be determined for the other quadrants. You should learn this table or learn how to derive these values from triangles (see Lesson 3, Facts 1 and 2).

The following table summarizes the trigonometric values for the Quadrantal angles:

Degrees	0°	30°	45°	60°	90°
Arc Length	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Unit Circle

Definitions



For any point P(x,y) on the unit circle (radius = 1) with an arc length of s in standard position.

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y} = \frac{1}{\tan\theta}$$

Solving Equations

The equations $y = \sin\theta$, $y = \cos\theta$, $y = \tan\theta$ have two solutions for the domain $0 < \theta < 2\pi$, or the domain $0° \le \theta < 360°$.

The equations $y = \sin n\theta$, $y = \cos n\theta$, $y = \tan n\theta$, where n is an integer, have $2n$ solutions for the domain $0° \le \theta < 360°$ or the domain $0° \le \theta < 2\pi$.

To solve the equations $y = \sec\theta$, $y = \csc\theta$, $y = \cot\theta$, use the reciprocals: $\cos\theta = \frac{1}{y}$, $\sin\theta = \frac{1}{y}$, $\tan\theta = \frac{1}{y}$

Some more information can be found in Michelson's book, *Theory and Problems for Senior High Math,* on pages 47–49, #1–7, and further problems #1–5 on pages 52–53.

Module 2, Section 1

Review

1. Find the exact value for each expression.

   a) $12 \sin 30^\circ \cos 60^\circ$

   b) $\dfrac{\tan \dfrac{\pi}{4}}{\sin\left(-\dfrac{5\pi}{6}\right)}$

   c) $\csc 210^\circ - 2\cot\left(-\dfrac{3\pi}{4}\right)$

   d) $\sin^2 \dfrac{3\pi}{2} - \cos^2 \pi$

   e) $2\sec^2 \dfrac{5\pi}{4} - \cot^2 30^\circ$

   f) $\cos 1235\pi + \sin 255\pi$

2. Solve the following equations over the interval $0 \le \theta < 2\pi$. Do these questions without the use of a calculator. All answers must be ***exact*** special values.

   a) $\sin\theta = -\dfrac{1}{\sqrt{2}}$

   b) $\tan\theta = -1$

   c) $\cos\theta = \dfrac{1}{2}$

   d) $\sin\theta = -\dfrac{1}{2}$ and $\tan\theta > 0$

   e) $\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\csc\theta < 0$

   f) $2\cos\theta = \sqrt{3}$

   g) $2\cos^2\theta + \cos\theta = 0$

   h) $4\cos^2\theta = 1$

   i) $2\cot\theta + 2\sqrt{3} = 0$

   j) $2\cos^2\theta + \cos\theta = 1$

   k) $\tan^2\theta - \tan\theta = 0$

3. Convert each degree measure into radian measure.

   a) $45°$

   b) $-150°$

   c) $500°$

   d) $600°$

4. Convert each radian measure into degree measure. Round off to one decimal place where necessary.

   a) $\frac{5\pi}{6}$ b) $-\frac{11\pi}{12}$

   c) 3.8 d) –0.5432

5. Solve each equation for x, where $0° \le x < 360°$, or for θ, where $0 \le \theta < 2\pi$.

   a) $2 \sin^2 \theta = \sin \theta$ b) $\sin x = \cos x$

   c) $\cos 2\theta = -1$ d) $2 \sin 2x = 1$

   e) $2 \csc \theta = -4$ f) $2 \tan^2 x - 1 = -\tan x$

   g) $2 \cos\left(\theta - \frac{\pi}{2}\right) - 1 = 0$ h) $\cos\left(\theta + \pi\right) = 0$

   i) $\cos x \tan x + \cos x = 0$

Check your answers in the Module 2 Answer Key.

Now do the Section Assignment which follows this section.

PRINCIPLES OF MATHEMATICS 12

Section Assignment 2.1

General Instructions for Section Assignments

These instructions apply to all the section assignments, but will not be reprinted each time. Remember them for future sections.

(1) Treat this assignment as a test, so do not refer to your module or notes or other materials. A graphing calculator is required.

(2) Where questions require computations or have several steps, showing these can result in part marks for some exercises. Steps must be neat and well-organized, however, or the instructor will only consider the answer.

(3) Always read the question carefully to ensure you answer what is asked. Often unnecessary work is done because a question has not been read correctly.

(4) Always clearly underline your final answer so that it is not confused with your work.

Section Assignment 2.1

Trigonometric Functions

Total Value: 48 marks
(Mark values in margins)

(11 marks)

1. Fill in the blanks with the correct response.

a) If $\sin\theta < 0$ and $\sec\theta > 0$, then θ terminates in Quadrant: _____

b) The exact value of $\sec\frac{7\pi}{6}$ is: _____

c) A 2π cm wire is stretched along a circle of radius 4 cm. The degree measure of the central angle subtended by this wire arc is: _____

d) Find the smallest possible angle in standard position of 4400°. _____

e) What is the reference angle for 4400°? _____

f) What is the reference angle for $\frac{27\pi}{5}$? _____

g) If $0 < \theta < \pi$ and $\cos\theta = 0.5$, then the value of $\cos(2\theta)$ is: _____

h) The value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, in degree measure, is: _____

i) The value of $16^{\cos\frac{\pi}{3}}$ is: _____

j) The exact value of $\cot\left(\frac{5\pi}{6}\right)$ is: _____

k) What is a coterminal angle with 170°? _____

2. Find the value of $\sin^3\left(-\frac{\pi}{2}\right)+2\cos^2\frac{\pi}{4}$. *(4)*

3. a) Evaluate tan 7.5. *(1)*

b) Evaluate exactly $\cos\left(-\frac{45\pi}{4}\right)$. *(1)*

4. As the time changes from 4:10 p.m. to 6:45 p.m. on the analog clock, determine the change in radian measure of the minute hand. *(2)*

5. Solve each equation for x, where $0 \leq x < 2\pi$.

(4) a) $2 \sin x + 1 = 0$

(5) b) $\cos x(2 \sin x - 1) = 0$

(4) c) $2 \cos 2x = 1$

(4) d) $\tan^2 x = 1$

6. Solve each equation or inequality for x:

a) $\sin\left(x+\frac{\pi}{2}\right)=\frac{1}{2}$ where $0 \le x < 2\pi$ *(4)*

b) Solve for $0° \le x < 360°$ *(4)*

$3 \tan 2x = 1$

c) $2 \sec x = 3$, where $0° \le x < 360°$ *(4)*

Total: 48 marks